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Erratum

Miscellany on traces in ℓ-adic cohomology: a survey

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Theorem 2.3 is correct as stated. However, in Lemma 7.1, in addition to the other hypotheses, S should be assumed geometrically unibranch. Otherwise the implication (1) $\Rightarrow$ (2) may not hold. Indeed, Gabber gave the following example. Let S be the $\mathbb{F}_p$-scheme obtained from the affine line over $\mathbb{F}_p$ by identification of the points 0 and 1. By definition, $S = \operatorname{Spec} A$, where A is the subring of $\mathbb{F}_p[t]$ consisting of the polynomials P such that $P(0) = P(1)$: S is the cubic in $\mathbb{A}^2_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x,y]$ with equation $x^3 - y^2 + xy = 0$ (an isomorphism $\mathbb{F}_p[x,y]/(x^3 - y^2 + xy) \xrightarrow{\sim} A$ is given by $x \mapsto t^2 - t, y \mapsto t(t^2 - t)$). Let Z be the connected étale cover of degree 2 of S which splits over the normalization of S. With the notations of Lemma 7.1, take $a = 2$ and $n \geq 1$ such that $\ell^n > 2$. Let $g \in \pi_1(S, \overline{\eta})$ correspond to the involution permuting the sheets of Z. Then $|Z(s)| = a = 2$ for every point s of S with value in a finite field. However, $\operatorname{Tr}(g, H^*_c(Z_{\overline{\eta}}, \mathbb{Q}_\ell)) = 0$, and $0 \neq 2$ in $\mathbb{Z}/\ell^n\mathbb{Z}$. So (1) does not imply (2) in this case. If S is assumed geometrically unibranch, then the proof given in *loc. cit.* is correct (the extra assumption is

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used in the application of Čebotarev's theorem). On the other hand, the proof of 2.3 given in 7.2 is correct, as the implication (1) $\Rightarrow$ (2) of 7.1 is used only in the case where S is normal.

In 7.1, for a $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$ on S, *lisse* is tacitly assumed to mean that $\mathscr{F}$ corresponds to a continuous representation of $\pi_1(S, \overline{\eta})$ in a finite dimensional $\mathbb{Q}_\ell$-vector space, in other words, $\mathscr{F} = \mathscr{F}_0 \otimes \mathbb{Q}_\ell$, for a lisse $\mathbb{Z}_\ell$-sheaf $\mathscr{F}_0$ on S. It is well known, however, that if S is not geometrically unibranch, there may exist $\mathbb{Q}_\ell$-sheaves on S which are lisse in the sense of [D2, 1.1.1], i.e., are *étale locally* of the form $\mathscr{F}_0 \otimes \mathbb{Q}_\ell$ for a lisse $\mathbb{Z}_\ell$-sheaf $\mathscr{F}_0$, but are not globally of this form (contrary to what is suggested in [D2, 1.1.6]). I learnt the following example from Gabber. Take $S = \operatorname{Spec} A$ as above. Let Y be the S-scheme deduced from $(\mathbb{A}^1 \times \mathbb{G}_m)_{\mathbb{F}_p}$ by identifying $\{0\} \times x$ with $\{1\} \times x^\ell$ (in other words, $Y = \operatorname{Spec} B$, where B is the subring of $\mathbb{F}_p[t, s, s^{-1}]$ consisting of the elements P such that $P(0, s) = P(1, s^\ell)$). Let $f : Y \to S$ be the projection; this is a morphism of finite type, with fibers isomorphic to $\mathbb{G}_m$. The $\mathbb{Q}_\ell$-sheaf $\mathscr{F} := R^2 f_! \mathbb{Q}_\ell$ on S is lisse of rank 1 in the sense of [D2, 1.1.1], but does not come from a lisse $\mathbb{Z}_\ell$-sheaf on S (if $\pi : \mathbb{A}^1_{\mathbb{F}_p} \to S$ is the normalization of S, and $i : \{0\} \to S$ the inclusion of the double point, $\mathscr{F}$ is the kernel of the map $\pi_* \mathbb{Q}_\ell(-1) \to i_* \mathbb{Q}_\ell(-1)$ defined by $(1, -\ell)$ at the double point).

page 134, line 9: A is not the sub-k-algebra B of K generated by Y and XY^{-n} for $n \geq 1$, but a localization of B. This error, however, is of no consequence for the rest of the remark.

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